Algorithm for recognizing rank-width at most k and the well-quasi-ordering of the vertex-minor relation

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Algorithms

For fixed k, we find an **fixed-parameter-tractable** algorithm that

- confirms that rank-width$> k$, or
- outputs the rank-decomposition of width $\leq 3k + 1$.

Consequence: We don't have to require a k-expression as an input to algorithms using it.

(Joint work with Paul Seymour)

Well-Linkedness and Rank-width

$A \subseteq V$ is called **well-linked** iff for any partition (X, Y) of A,

$$X \subseteq Z \subseteq V \setminus Y \quad \Rightarrow \quad \mathrm{cutrk}(Z) \geq \min(|X|, |Y|).$$

Thm. (O., Seymour)

1. If $\exists$ **well-linked set A of size k**, then $\mathrm{rwd}(G) \geq k/3$. **Otherwise**, $\mathrm{rwd}(G) \leq k$

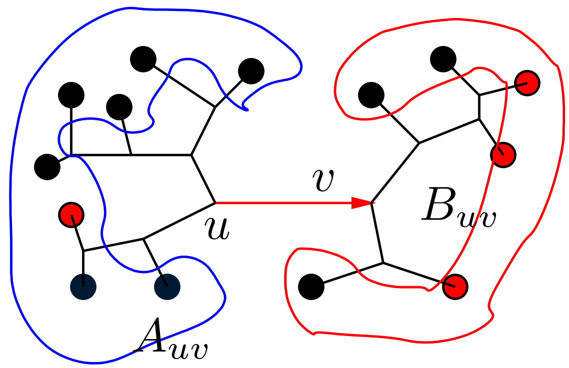
2. $\exists$ a poly-time algorithm that constructs the rank-decomp. of width $\leq k$ or finds a well-linked set of size k.

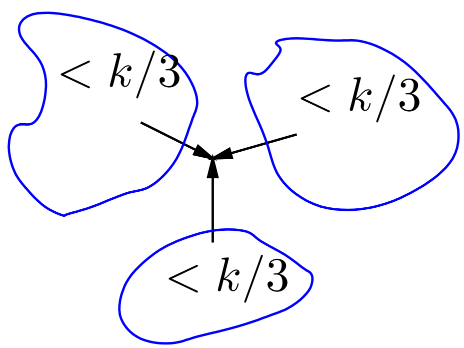
$\Rightarrow$ poly-time algorithm to confirm $\mathbf{rwd}(G) > k$ or $\mathbf{rwd}(G) \leq 3k+1$ and output its rank-decomposition of width $\leq 3k+1$.

Easy direction — standard trick

Lemma. Every directed tree has a node that all incident edges are incoming.

Claim: If $\exists$ **well-linked set** A of size k, then $\mathbf{rwd}(G) \geq k/3$.





Suppose $rwd(G) < k/3$.
Let T be the rank-decomp. of width $< k/3$.
If $\exists\, uv \in E(T)$, $|A_{uv} \cap A|, |B_{uv} \cap A| \geq k/3$, then $\mathrm{cutrk}(A_{uv}) \geq \min(|A_{uv} \cap A|, |B_{uv} \cap A|) \geq k/3$, **because A is well-linked**.
For $\forall uv \in E(T)$,
either $|A_{uv} \cap A| < k/3$ or $|B_{uv} \cap A| < k/3$.

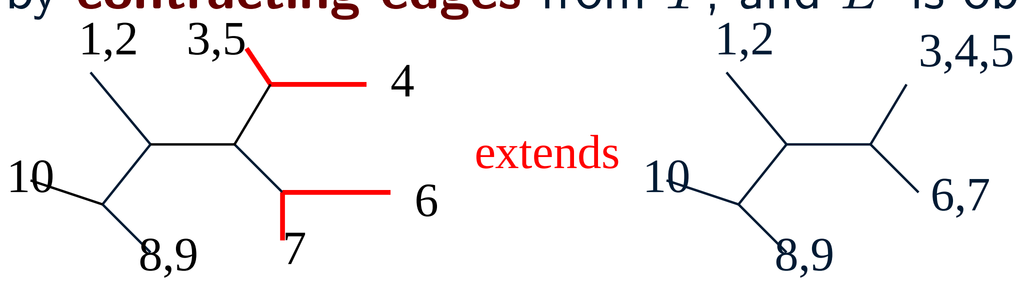
Direct $u \to v$ if $|A_{uv} \cap A| < k/3$.
$|A| < k/3 + k/3 + k/3$. Contradiction.

Another direction

Claim: Suppose there is no well-linked set of size k. Then, every **partial rank-decomposition** of width$\leq k$ can be **extend**ed to a rank-decoomposition of width$\leq k$,

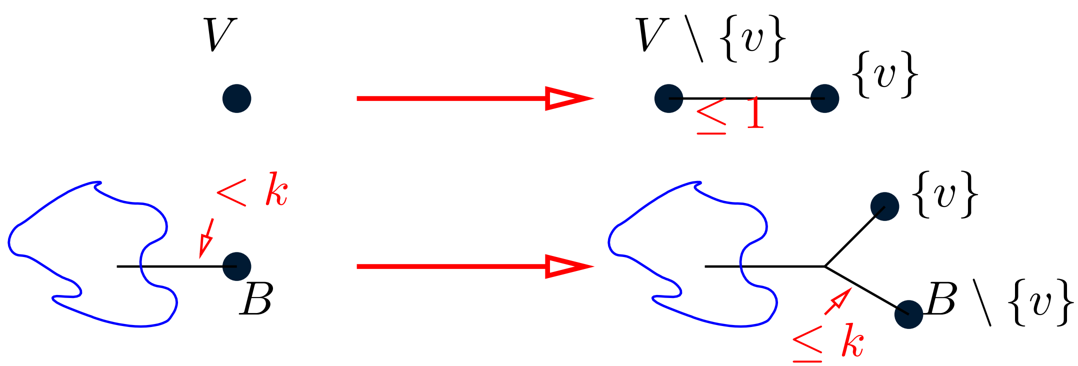
- Rank-decomposition of G: cubic tree T with a bijective function $L : V(G) \to \{\text{leaves of T}\}$.
- **Partial rank-decomposition** of G: cubic tree T with a **surjective** function $L : V(G) \to \{\text{leaves of T}\}$.

Say a partial rank-decomposition (T, L) extends (T', L') if T' is obtained by **contracting edges** from T, and L' is obtained from L canonically.



Another direction — cont.

At each step, we try to increase the number of leaves.

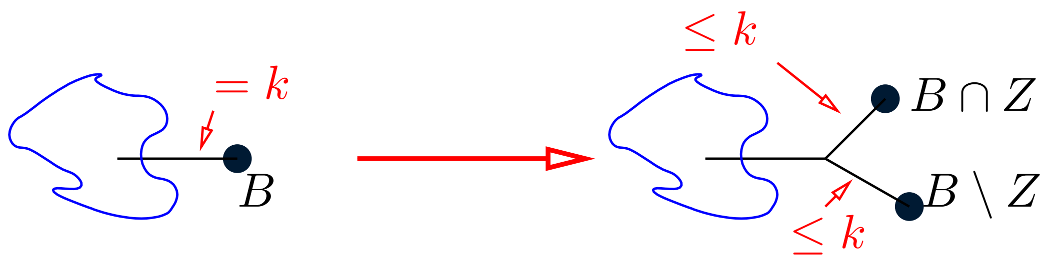


because if $\mathrm{cutrk}_G(B) < k$, then

$$\mathrm{cutrk}_G(B \setminus \{v\}) \leq \mathrm{cutrk}_G(B) + \mathrm{cutrk}_G(\{v\}) \leq k$$

by submodularity.

Last case to consider



Pick $X \subseteq V \setminus B$ "basis" such that $\mathrm{cutrk}^*(X, B) = |X| = k$.
X is not well-linked; $\exists Z$ such that $\mathrm{cutrk}(Z) < \min(|X \cap Z|, |X \setminus Z|)$. By submodularity,

$$\mathbf{cutrk}(\boldsymbol{Z}) + \mathbf{cutrk}(\boldsymbol{B}) \geq \mathbf{cutrk}(\boldsymbol{Z} \cap \boldsymbol{B}) + \mathbf{cutrk}(\boldsymbol{Z} \cup \boldsymbol{B}).$$

$\mathrm{cutrk}(Z \cup B) = \mathrm{cutrk}^*(V \setminus B \setminus Z, B) \geq \mathrm{cutrk}^*(x \setminus Z, B) = |X \setminus Z|$.
Therefore, $k \geq \mathrm{cutrk}(B) \geq \mathrm{cutrk}(Z \cap B)$.
$B \cap Z \neq \emptyset$; suppose not. Then, $\mathrm{cutrk}(Z) \geq \mathrm{cutrk}^*(X \cap Z, B) = |X \cap Z|$.

Time complexity

Let γ be the running time of matrix rank calculation.

We use the **submodular function minimization algorithm** by [Iwata et al., 2001], whose runnning time is $O(n^5\delta \log M)$. M is the maximum value of the submodular function and δ is the running time of the submodular function.

Job	Time
Find a "basis" X	$O(n\gamma)$
Find Z	$O(2^{k-1}(n^5\gamma \log n))$

$O(n(n\gamma + 2^{k-1}n^5\gamma \log n)) = O(n^6\gamma \log n)$.

For rank-width: $\gamma = O(n^3) \Rightarrow$ Running time: $\boldsymbol{O(n^9 \log n)}$.
Fixed-parameter-Tractable!

Well-quasi-ordering by the vertex-minor rel.

- The **graphs of tree-width**$\leq k$ are well-quasi-ordered by the minor relation. [Robertson and Seymour, 1990]
- The **matroids of branch-width**$\leq k$ are well-quasi-ordered by the minor relation. [Geelen et al., 2002]
- The **graphs of rank-width**$\leq k$ are well-quasi-ordered by the **vertex-minor** relation.

Well-quasi-ordering

Thm. The graphs of rank-width$\leq k$ are **well-quasi-ordered** by the vertex-minor relation.

In other words:

- Let k be **fixed**.
- Let G_1, G_2, G_3, $\ldots$ be any **infinite** sequence of graphs such that G_i has rank-width$\leq k$.
- Then, there is $i < j$ such that G_i is isomorphic to a **vertex-minor** of G_j.

Note: Instead of saying that H is isomorphic to a vertex-minor of G, we sometimes say H is a vertex-minor of G.

Four Ingredients to prove wqo

- **Extension of Menger's theorem**
  $X, Y \subseteq V(G)$, $X \cap Y = \emptyset$. Then,
  $\min\{\text{CUT-RANK}_G(Z) : X \subseteq Z \subseteq V - Y\}$
  $= \max\{\text{CUT-RANK}_H(X) : H = \text{vertex-minor of } G, V(H) = X \cup Y\}$

- **If rank-width of G is n, then there is a linked rank-decompositon of width n.** [Geelen et al., 2002] cf. [Thomas, 1990]
  For any e, f in the rank-decomposition T, any vertex partition separating e, f has cut-rank $\geq$ min cut-rank of an edge in the path from e to f in T.
- **Robertson and Seymour's "Lemma on trees"** [Robertson and Seymour, 1990]
- **Isotropic system** [Bouchet, 1987]

Scrap

Let V be a finite set.
$S = (V, L)$ is an isotropic system
if L is a totally isotropic subspace of K^V and $\dim(L) = |V|$.

$P = (V, L, B)$ is a scrap if

- L is a totally isotropic subspace of K^V,
- B is an **ordered** set (sequence) and a **basis of $\boldsymbol{L^\perp/L}$**.

$\dim(L^\perp/L) = (2|V| - \dim(L)) - \dim(L) = 2(|V| - \dim(L))$. If $B = \emptyset$, then (V, L) is an isotropic system.

$P_1 = (X, L', B')$ is a minor of P if $X = V \setminus \{v_1, v_2, \ldots, v_k\}$, $L' = L|_{x_1}^{v_1}|_{x_2}^{v_2} \cdots |_{x_k}^{v_k}$, and $|B'| = |B|$ and B' is obtained naturally from B by $\cdots$.

Very Rough Sketch of Proof

Suppose G_1, G_2, ... is a sequence of graphs of rank-width at most k and there is no $i < j$ such that G_i is isomorphic to a vertex-minor of G_j.

Let F be an infinite forest such that each component is the linked rank-decomposition of G_i. We attach the root vertex to each component. For an edge e, let $l(e)$, $r(e)$ be the left/right child edge incident to e. We assign a **scrap** — a piece of information — to each edge of F and define a relation $\leq$ on the set of edges of F. We make a scrap of e is a **sum** of scraps of $l(e)$ and $r(e)$.

By applying lemma on trees, we get a sequence e_0, e_1, ... of edges such that $\{e_0, e_1, \ldots\}$ is an antichain and $l(e_0) \leq l(e_1) \leq l(e_2) \leq \cdots$ and $r(e_0) \leq r(e_1) \leq r(e_2) \leq \cdots$.

The number of ways to **sum** 2 scraps is finite $\Rightarrow \exists i < j$, $e_i \leq e_j$. Contradiction.

Excluded vertex-minors for rank-width$\leq k$

G is an **excluded vertex-minor** for a class of graphs of rank-width$\leq k$ if

- Rank-width of $G > k$
- Every proper vertex-minor of G has rank-width$\leq k$.

Cor. For fixed k, there are **only finitely many excluded vertex-minors** for a class of graphs of rank-width $\leq k$.

Proof. An excluded vertex-minor has rank-width $k+1$. Let E be the set of excluded vertex-minors. E is well-quasi-ordered by the vertex-minor relation. But, no excluded vertex-minor contains another. So, E is finite. $\square$

Note: The above corollary has an elementary proof.
Cor. For fixed k, **"Is rank-width$\leq k$?"** is **NP$\cap$ coNP**.

Rank-width$\leq k$ is now in P

Thm (Courcelle, O.). Let H be a fixed graph. There is a modulo 2 counting monadic second order logic formula, describing H is a vertex-minor of an input graph.

Monadic second order logic formula $+$ IsEven(U)

Proof. Graphs $\Rightarrow$ Isotropic system$\Rightarrow$ Minor of isotropic system $\Rightarrow$ Vertex-minor $\square$

Cor. For fixed k, rank-width $\leq k$ is decidable in $O(n^9 \log n)$.

Rank-width≤ 1 graphs

Thm. • Adding a pendent vertex or a twin doesn't change the rank-width of a graph.

• If G has rank-width≤ 1 and $|V(G)| \geq 2$, then there always is a pair v, w of vertices such that one of the following is true:

1. v is a twin of w or
2. v is a pendant vertex of w or an isolated vertex.

Prop. The rank-width of $G \leq 1$ if and only if G is a **distance-hereditary graph**.

Remind $\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2^{1+\mathrm{rwd}(G)} - 1$.

Cor. The clique-width of a distance-hereditary graph is at most 3.

Cor. Distance-hereditary graphs are well-quasi-ordered by the vertex-minor relation.

Open problems — "Graph Vertex-Minor Project"

- Generalize the following to graphs with rank-width:

  [Geelen et al., 2003]
  Let k be a constant. If branch-width of a binary matroid is sufficently large, then it contains a cycle matroid of the $k \times k$ grid.

  Conjecture: Let H be a circle graph (intersection graph of chords). Graphs with no vertex-minor isomorphic to H have rank-width $\leq f(H)$.
- How to avoid using the general submodular function minimization algorithm? That will improve the time complexity.
  If G is bipartite, then it can be done by matroid intersection theorem.

Questions?

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