

A PROOF OF THE CYCLE DOUBLE COVER CONJECTURE BY OPENAI: AN EXPOSITION

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ABSTRACT. The cycle double cover conjecture states that every bridgeless graph has a list of cycles such that every edge is in exactly two of them. In July 2026, OpenAI announced a proof. This exposition presents the proof with slight modifications intended to make it more accessible.

1. INTRODUCTION

A graph is *Eulerian* if every vertex has even degree. A *cycle* of a graph is a 2-regular connected subgraph.

A *cycle double cover (CDC)* is a list of cycles such that every edge is in exactly two of them. Since every Eulerian subgraph can be decomposed into pairwise edge-disjoint cycles, for the existence of a cycle double cover, it is equivalent to define a cycle double cover as a list of Eulerian subgraphs such that every edge is in exactly two of them.

A *bridge* of a graph is an edge whose deletion increases the number of components. Observe that a bridge is not in any cycle. Thus to have a cycle double cover, it is necessary to be bridgeless.

This note is to discuss a proof of the cycle double cover conjecture, announced by OpenAI [15].

Theorem 1. *Every bridgeless graph has a cycle double cover.*

According to Zhang [22, 23], this conjecture was independently stated by Seymour [17] for all bridgeless graphs and Szekeres [19] for all bridgeless cubic graphs. An equivalent version was stated by Itai and Rodeh [8].

One of the motivations to study the cycle double cover conjecture was the following stronger conjecture on embeddings of graphs. An *embedding* of a graph on a surface is a drawing of G on the surface such that no two edges cross. An embedding is called a *2-cell embedding* if every face is homeomorphic to an open disk. A *strong embedding* of a graph G on a surface Σ is a 2-cell embedding such that every face boundary is a cycle of G .

Conjecture 2 (Strong Embedding Conjecture). *Every 2-connected graph has a strong embedding on a surface.*

Conjecture 2 implies Theorem 1, just by first reducing to 2-connected graphs and taking all face boundaries. Conjecture 2 remains open. For more about the cycle double cover conjecture, survey papers and books were written by Jaeger [10], Zhang [22, 23, 21], and Chan [4].

The proof presented in this note is due to GPT 5.6 announced by OpenAI [15] in July 2026. The proof was modified mainly in two parts.

- The choice of two values of $\mathbb{F}_2^3$ at each edge is now symmetric. We do not choose a particular edge at each vertex.

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- The linear algebra proof is done without using dual vector spaces and is based on an elementary fact that the column space of a matrix is the orthogonal complement of its left null space.

Our goal is to present the whole proof in mostly self-contained manner, which would be suitable to teach the proof to advanced undergraduate students. We will only assume two theorems: Fleischner's splitting lemma and a theorem on packing spanning trees. We allow parallel edges and loops in graphs.

The last section will discuss more conjectures related to the cycle double cover conjecture.

Statement of AI use. Parts of the revision were carried out with GPT 5.6 under the author's guidance.

2. SPLITTING LEMMA

For an integer k , a graph G with at least two vertices is k -edge-connected if for every $X \subseteq V(G)$ with $0 < |X| < |V(G)|$, G has at least k edges having one end in X and the other end in $V(G) \setminus X$. We omit the proof of the following lemma.

Lemma 3 (Fleischner's splitting lemma [6, Lemma III.26]). *Let G be a 2-edge-connected graph, let v be a vertex of degree at least four, and let e_1, e_2, e_3 be distinct edges incident with v . For each i , let v_i be the other end of e_i . If $G - \{e_1, e_2, e_3\}$ is connected, then at least one of*

$$(G - \{e_1, e_2\}) + v_1v_2 \quad \text{and} \quad (G - \{e_2, e_3\}) + v_2v_3$$

is 2-edge-connected.

If G is 3-edge-connected and $d(v) \geq 4$, then the three edges in Lemma 3 can always be chosen. If $G - v$ is connected, a spanning tree of $G - v$ together with one edge incident with v is a spanning tree of G that omits at least three edges incident with v . If $G - v$ is disconnected, every component of $G - v$ sends at least three edges to v . Taking a spanning tree in each component and one edge from that component to v again gives a spanning tree of G that omits at least three edges incident with v . Any three such omitted edges have the required property.

3. MINIMUM COUNTEREXAMPLE TO THE CYCLE DOUBLE COVER CONJECTURE

We say a graph is *cubic* if every vertex has degree 3. The following reduction is well known, see Zhang [22].

Proposition 4. *The cycle double cover conjecture is true if and only if it is true for cubic 3-edge-connected graphs.*

Proof. Suppose the cycle double cover conjecture is false, and let G be a counterexample with $|E(G)|$ minimum. Deleting a loop and covering it twice shows that G has no loops, and considering components shows that G is connected.

We first show that G is 3-edge-connected. Otherwise it has a 2-edge-cut $\{e_1, e_2\}$. Contract e_1 , and let w be the resulting vertex. Contraction cannot create a bridge, so G/e_1 is a smaller bridgeless graph and hence has a cycle double cover $\mathcal{C}$ by minimality.

Consider the two shores of $G - \{e_1, e_2\}$. If e_2 is not a loop in G/e_1 , a cycle of $\mathcal{C}$ containing e_2 passes through w from one shore to the other, and a cycle not containing e_2 cannot do so. Split w back into the ends of e_1 and insert e_1 into the two cycle occurrences containing e_2 ; all other cycles lift unchanged. If e_2 is a loop in G/e_1 , replace its two occurrences instead by two copies of the 2-cycle formed by e_1 and e_2 . In either case we obtain a cycle double cover of G , a contradiction. Thus G is 3-edge-connected.

Suppose that $d(v) \geq 4$ for some vertex v . Choose e_1, e_2, e_3 as above and apply Lemma 3. Let H be the resulting 2-edge-connected graph with $|E(H)| < |E(G)|$, so by minimality it has a

cycle double cover. Replacing each occurrence of its new edge by the corresponding two-edge path through v , and decomposing closed trails into cycles if necessary, gives a cycle double cover of G , a contradiction. Hence G is cubic. $\square$

4. DISJOINT SPANNING TREES

Theorem 5 (Tutte [20] and Nash-Williams [14]). *A graph G has t edge-disjoint spanning trees if and only if for every partition $V_1, \dots, V_m$ of $V(G)$ with nonempty parts, there are at least $t(m-1)$ edges joining distinct parts.*

One of the easiest ways to prove [Theorem 5](#) is to use the matroid union theorem. There are purely graph-theoretic proofs known [13]. We omit its proof.

Corollary 6. *If G is $2k$ -edge-connected, then G has k edge-disjoint spanning trees.*

Proof. Let $V_1, \dots, V_m$ be a partition of $V(G)$ with nonempty parts. Each set V_i has at least $2k$ edges in its edge boundary. When summing these for all i , we count each crossing edge twice. Therefore, the number of edges joining distinct parts of $V_1, \dots, V_m$ is at least km , which is at least $k(m-1)$. $\square$

Jaeger [12, 9] proved that every 2-edge-connected graph has a nowhere-zero $\mathbb{Z}_2^3$ -flow by using [Theorem 5](#). Here I would like to write its main idea, without mentioning the nowhere-zero 8-flows.

Lemma 7. *Let G be a 3-edge-connected graph. Then there are three spanning trees T_1, T_2, T_3 such that $E(T_1) \cap E(T_2) \cap E(T_3) = \emptyset$.*

Proof. Let G' be a graph obtained from G by adding a parallel copy of every edge so that $|E(G')| = 2|E(G)|$. Then G' is 6-edge-connected. By [Corollary 6](#), G' has three edge-disjoint spanning trees T'_1, T'_2, T'_3 . Those three spanning trees correspond to spanning trees T_1, T_2, T_3 of G such that no edge belongs to all three of them at the same time. $\square$

For a vertex v of a graph G , let $\delta(v) = \{e \in E(G) : v \text{ is an end of } e\}$.

Lemma 8. *Let T be a spanning tree of a graph G . Then there is a set F of edges of G such that $E(G) \setminus E(T) \subseteq F$ and $|F \cap \delta(v)|$ is even for every vertex v of G .*

Proof. For every $e \in E(G) \setminus E(T)$, there is a cycle C_e in $T + e$. Define

$$F = \bigtriangleup_{e \in E(G) \setminus E(T)} E(C_e),$$

which is the set of all edges in an odd number of C_e 's. Now the conclusion follows. $\square$

For an abelian group Γ , a *nowhere-zero Γ -flow* of a graph G is a function $\phi : E(\vec{G}) \rightarrow \Gamma$ for an orientation $\vec{G}$ of G such that $\sum_{e \in \delta^+(v)} \phi(e) = \sum_{e \in \delta^-(v)} \phi(e)$ for every vertex v of G and $\phi(e) \neq 0$ for all edges e of G . If $x = -x$ for every $x \in \Gamma$, then there is no need to specify an orientation, as in our setting.

Theorem 9 (Jaeger [12, 9]). *Every 2-edge-connected graph has a nowhere-zero $\mathbb{Z}_2^3$ -flow.*

We only need its special case.

Lemma 10 (Jaeger [12, 9]; weaker). *Every 3-edge-connected graph has a nowhere-zero $\mathbb{Z}_2^3$ -flow.*

Proof. By [Lemma 7](#), choose spanning trees T_1, T_2, T_3 such that $E(T_1) \cap E(T_2) \cap E(T_3) = \emptyset$. For each $i \in \{1, 2, 3\}$, [Lemma 8](#) gives a set $F_i \subseteq E(G)$ containing $E(G) \setminus E(T_i)$ and having even degree at every vertex. Let ϕ_i be the characteristic function of F_i and define

$$\phi : E(G) \longrightarrow \mathbb{Z}_2^3, \quad \phi(e) = \begin{pmatrix} \phi_1(e) \\ \phi_2(e) \\ \phi_3(e) \end{pmatrix}.$$

Each coordinate sums to zero at every vertex. Moreover, every edge e is omitted by some T_i and hence belongs to F_i . Thus $\phi(e) \neq 0$, so ϕ is a nowhere-zero $\mathbb{Z}_2^3$ -flow. $\square$

5. BUILDING A CYCLE DOUBLE COVER

Lemma 11 (Two-element edge labels). *Let G be a loopless graph and let Γ be a finite set. Suppose that every edge e is assigned a two-element set $P_e \subseteq \Gamma$ such that*

$$|\{e \in \delta(v) : s \in P_e\}| \text{ is even} \quad \text{for all } v \in V(G) \text{ and } s \in \Gamma.$$

Then G has a cycle double cover.

Proof. For each $s \in \Gamma$, let

$$H_s = \{e \in E(G) : s \in P_e\}.$$

Every vertex has even degree in H_s , so H_s is Eulerian. Every edge belongs to exactly two of the graphs H_s , since P_e has two elements. Thus all the H_s form a cycle double cover of G . $\square$

Remember that for a subspace W' of a vector space W and an element x of W , we have $x + W' = \{x + v : v \in W'\}$. For a vector x in W , let $\langle x \rangle$ denote the span of x . Let $\mathbb{F}_2 = \{0, 1\}$ be a field with two elements.

Lemma 12 (Flow lifting). *Let G be a cubic loopless graph. Let $\Gamma = \mathbb{F}_2^3$. Let $\phi : E(G) \rightarrow \Gamma$ be a function such that $\phi(e) \neq 0$ for every edge e of G and $\sum_{e \in \delta(v)} \phi(e) = 0$ for every vertex v of G . If there are $t_v \in \Gamma$ for all $v \in V(G)$ such that for every edge $e = uv$ and all $f_u \in \delta(u) \setminus \{e\}$ and $f_v \in \delta(v) \setminus \{e\}$,*

$$t_u + t_v \in (\phi(f_u) + \phi(f_v)) + \langle \phi(e) \rangle,$$

then G has a cycle double cover.

Proof. Note that if $e, f, g \in \delta(v)$ are distinct edges incident with v , then $\phi(e) + \phi(f) + \phi(g) = 0$ and therefore $\phi(f) + \langle \phi(e) \rangle = \phi(g) + \langle \phi(e) \rangle$. Thus the choice of f_u or f_v does not change the value.

For each edge $e = uv$ of G , let $P_e = t_u + \phi(f_u) + \langle \phi(e) \rangle$. By the choice of t_v , we have $P_e = t_v + \phi(f_v) + \langle \phi(e) \rangle$ and therefore P_e is well defined. Furthermore, $|P_e| = 2$.

Observe that for a vertex v with three incident edges e, f, g ,

$$\begin{aligned} P_e &= \{t_v + \phi(f), t_v + \phi(g)\}, \\ P_f &= \{t_v + \phi(e), t_v + \phi(g)\}, \\ P_g &= \{t_v + \phi(e), t_v + \phi(f)\}. \end{aligned}$$

Now, we apply [Lemma 11](#). $\square$

6. SOME LINEAR ALGEBRA

For an $m \times n$ matrix A over a field $\mathbb{F}$, we write $\mathcal{C}(A) = \{Ax \in \mathbb{F}^m : x \in \mathbb{F}^n\}$ for the *column space* and $\ker(A^\top) = \{y \in \mathbb{F}^m : A^\top y = 0\}$ for the *left nullspace*.

Lemma 13 (Column space and left nullspace). *Let $\mathbb{F}$ be a field and let A be an $m \times n$ matrix over $\mathbb{F}$. With respect to the standard dot product on $\mathbb{F}^m$,*

$$\mathcal{C}(A) = \ker(A^\top)^\perp.$$

Proof. If $\mathbf{y} \in \ker(A^\top)$, then $\mathbf{y}^\top A = 0$, so $\mathbf{y}$ is orthogonal to every column of A . Thus $\mathcal{C}(A) \subseteq \ker(A^\top)^\perp$. The two spaces have the same dimension, namely $\text{rank } A$, and hence they are equal. $\square$

Proposition 14 (Elementary consistency criterion). *Let $\mathbb{F}$ be a field, let $n \geq 2$ be an integer, and put $W = \mathbb{F}^n$ with the standard dot product. Let G be a graph. For every edge e , let p_e be a nonzero vector in W and let $d_e \in W$. Then the following are equivalent.*

(i) *There are vectors $t_v \in W$ such that*

$$t_u + t_v \in d_e + \langle p_e \rangle \quad \text{for all } e = uv.$$

(ii) *For every family $(h_e)_{e \in E(G)}$ of vectors in W , if*

$$h_e \cdot p_e = 0 \quad \text{for all } e \in E(G) \quad \text{and} \quad \sum_{e \in \delta(v)} h_e = 0 \quad \text{for all } v \in V(G),$$

$$\text{then } \sum_{e \in E(G)} h_e \cdot d_e = 0.$$

Proof. Suppose first that (i) holds, and let $(h_e)_{e \in E(G)}$ satisfy the two hypotheses in (ii). For each edge $e = uv$, choose $\lambda_e \in \mathbb{F}$ such that

$$t_u + t_v = d_e + \lambda_e p_e.$$

Since $h_e \cdot p_e = 0$, we have

$$\sum_{e \in E(G)} h_e \cdot d_e = \sum_{e=uv \in E(G)} h_e \cdot (t_u + t_v) = \sum_{v \in V(G)} \left(\sum_{e \in \delta(v)} h_e \right) \cdot t_v = 0.$$

Thus (ii) holds.

Conversely, suppose that (ii) holds. For each edge e , choose a basis $r_{e,1}, \dots, r_{e,n-1}$ of $\langle p_e \rangle^\perp = \{h \in W : h \cdot p_e = 0\}$. Let R_e be the $(n-1) \times n$ matrix whose i -th row is $r_{e,i}^\top$. Since $(\langle p_e \rangle^\perp)^\perp = \langle p_e \rangle$, condition (i) is equivalent to

$$R_e(t_u + t_v) = R_e d_e \quad \text{for all } e = uv.$$

Form the block matrix

$$A \in \mathbb{F}^{(n-1)|E(G)| \times n|V(G)|}$$

with block rows indexed by $e \in E(G)$ and block columns indexed by $v \in V(G)$, where

$$A_{e,v} = \begin{cases} R_e & \text{if } e \in \delta(v), \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbf{t} = (t_v)_{v \in V(G)}$ and $\mathbf{b} = (R_e d_e)_{e \in E(G)}$, then condition (i) is precisely the solvability of

$$A\mathbf{t} = \mathbf{b}.$$

By Lemma 13, it remains to show that $\mathbf{b}$ is orthogonal to the left nullspace $\ker(A^\top)$. Take $\mathbf{y} = (y_e)_{e \in E(G)} \in \ker(A^\top)$, where $y_e \in \mathbb{F}^{n-1}$, and put

$$h_e = R_e^\top y_e.$$

Then $h_e \in \langle p_e \rangle^\perp$, and the block of $A^\top \mathbf{y}$ indexed by a vertex v is $\sum_{e \in \delta(v)} h_e$. Hence $\sum_{e \in \delta(v)} h_e = 0$ for every vertex v . By (ii),

$$\mathbf{y}^\top \mathbf{b} = \sum_{e \in E(G)} y_e^\top R_e d_e = \sum_{e \in E(G)} h_e \cdot d_e = 0.$$

Thus $\mathbf{b} \in \ker(A^\top)^\perp = \mathcal{C}(A)$. Equivalently, $A\mathbf{t} = \mathbf{b}$ has a solution, and hence (i) holds. $\square$

7. PROOF FOR THE CUBIC 3-EDGE-CONNECTED GRAPHS

Proposition 15. *Let G be a cubic 3-edge-connected graph. Then G has a cycle double cover.*

Proof of Proposition 15. Put $\Gamma = \mathbb{F}_2^3$. Lemma 10 yields a nowhere-zero Γ -flow $\phi : E(G) \rightarrow \Gamma$.

If v is an end of an edge e , choose either edge $f \in \delta(v) \setminus \{e\}$ and put $c_{v,e} = \phi(f)$. The two choices differ by $\phi(e)$, so the coset $c_{v,e} + \langle \phi(e) \rangle$ is independent of the choice.

For every edge $e = uv$, put $d_e = c_{u,e} + c_{v,e}$. By Lemma 12, it is enough to show that there are $t_v \in \Gamma$ for all $v \in V(G)$ such that

$$(1) \quad t_u + t_v \in d_e + \langle \phi(e) \rangle \quad \text{for all } e = uv.$$

We apply Proposition 14 with $\mathbb{F} = \mathbb{F}_2$, $n = 3$, $W = \Gamma$, and $p_e = \phi(e)$. Consider any family $(h_e)_{e \in E(G)} \in \Gamma^{E(G)}$ satisfying

$$(2) \quad h_e \cdot \phi(e) = 0 \quad \text{for all } e \in E(G), \quad \sum_{e \in \delta(v)} h_e = 0 \quad \text{for all } v \in V(G).$$

It remains to prove that $\sum_e h_e \cdot d_e = 0$.

Fix a vertex v and write $\delta(v) = \{e_1, e_2, e_3\}$. The three values $\phi(e_i)$ are nonzero and pairwise distinct, and

$$\phi(e_1) + \phi(e_2) + \phi(e_3) = 0, \quad h_{e_1} + h_{e_2} + h_{e_3} = 0.$$

Put $\lambda = h_{e_2} \cdot \phi(e_1)$. Using $h_{e_i} \cdot \phi(e_i) = 0$ in the two displayed identities gives

$$h_{e_1} \cdot \phi(e_2) = h_{e_2} \cdot \phi(e_1) = h_{e_3} \cdot \phi(e_1) = \lambda.$$

If $\lambda = 1$, these equalities show that all three h_{e_i} are nonzero. If $\lambda = 0$, all three lie in the one-dimensional space $\langle \phi(e_1), \phi(e_2), \phi(e_3) \rangle^\perp$, and their sum is zero, so an even number of them are nonzero. Thus

$$\lambda = \sum_{e \in \delta(v)} \mathbf{1}_{h_e \neq 0}.$$

Here, $\mathbf{1}_{h_e \neq 0}$ is a function that has value 1 if $h_e \neq 0$ and 0 otherwise. Because changing the choice of $c_{v,e}$ adds $\phi(e)$ and $h_e \cdot \phi(e) = 0$, we may compute its dot product using either choice. Hence

$$(3) \quad \sum_{e \in \delta(v)} h_e \cdot c_{v,e} = h_{e_1} \cdot \phi(e_2) + h_{e_2} \cdot \phi(e_1) + h_{e_3} \cdot \phi(e_1) = \lambda = \sum_{e \in \delta(v)} \mathbf{1}_{h_e \neq 0}.$$

Since $d_e = c_{u,e} + c_{v,e}$, summing (3) over all vertices gives

$$\sum_{e \in E(G)} h_e \cdot d_e = \sum_{v \in V(G)} \sum_{e \in \delta(v)} h_e \cdot c_{v,e} = \sum_{v \in V(G)} \sum_{e \in \delta(v)} \mathbf{1}_{h_e \neq 0} = 0.$$

The last equality holds in $\mathbb{F}_2$, because every edge with $h_e \neq 0$ is counted at both ends. By Proposition 14, the required $(t_v)_{v \in V(G)}$ exist. This proves the claim. $\square$

8. COMPLETING THE PROOF

Theorem 1. *Every bridgeless graph has a cycle double cover.*

Proof. By Proposition 4, it is enough to prove it for cubic 3-edge-connected graphs. Proposition 15 gives the final contradiction. $\square$

9. MORE CONJECTURES

9.1. 5-cycle double cover conjecture. A k -cycle double cover (k -CDC) is a collection of at most k Eulerian subgraphs such that every edge is in exactly two of them.

The proof of [Theorem 1](#) also yields the following.

Theorem 16. *Every bridgeless graph has an 8-cycle double cover.*

Since the Peterson graph is not 3-edge-colorable, not all graphs have a 4-cycle double cover by the following theorem.

Theorem 17 (Jaeger [10]). *The following are equivalent for loopless cubic graphs G .*

- (i) G is 3-edge-colorable.
- (ii) G has a 3-cycle double cover.
- (iii) G has a 4-cycle double cover.

Proof. (iii)→(ii): If G has a 4-cycle double cover C_1, C_2, C_3, C_4 , then $C_1 \triangle C_2, C_1 \triangle C_3, C_1 \triangle C_4$ is a 3-cycle double cover.

(ii)→(i): Let C_1, C_2, C_3 be a 3-cycle double cover. Color each edge by the unique index $i \in \{1, 2, 3\}$ for which $e \notin C_i$. At every vertex, each C_i has degree two: its degree is even and the sum of these three degrees is six. Thus each C_i omits exactly one of the three incident edges. Since every edge is omitted by exactly one C_i , the three incident edges receive distinct colors. Hence this is a proper 3-edge-coloring.

(i)→(iii): Let M_1, M_2, M_3 be the three color classes of a proper 3-edge-coloring. They are perfect matchings forming a partition of $E(G)$. The Eulerian subgraphs

$$M_1 \triangle M_2, \quad M_2 \triangle M_3, \quad M_1 \triangle M_3$$

form a 3-cycle double cover: an edge in M_i belongs to precisely the two symmetric differences containing M_i . This is also a 4-cycle double cover. $\square$

There is a conjecture that one can reduce 8 to 5.

Conjecture 18 (Celmins [3], Preissmann [16]). *Every bridgeless graph has a 5-cycle double cover.*

9.2. Orientable cycle double cover conjecture. An *orientable cycle double cover* of a graph G is a list of Eulerian subgraphs, each equipped with a directed Eulerian orientation, such that every edge of G is used exactly once in each direction. An *orientable k -cycle double cover* of a graph G is an orientable cycle double cover with at most k Eulerian subgraphs.

Conjecture 19 (Archdeacon [1] and Jaeger [11]). *Every bridgeless graph admits an orientable 5-cycle double cover.*

Actually, this conjecture would imply the 5-flow conjecture of Tutte.

Lemma 20 (Lemma 13.1.5 in [22]). *If a graph G admits an orientable k -cycle double cover, then it has a nowhere-zero $\mathbb{Z}_k$ -flow.*

Proof. Pad the list with empty directed Eulerian subgraphs, and write it as $C_1, \dots, C_k$. Fix an arbitrary orientation of G . For each i , let χ_i be the signed characteristic function of C_i with respect to this orientation. Since C_i is directed Eulerian, χ_i is an integer flow. Therefore

$$\phi = \sum_{i=1}^k i\chi_i \pmod{k}$$

is a $\mathbb{Z}_k$ -flow. For each edge e , there are $i \neq j$ such that C_i and C_j use e in opposite directions. Hence $\phi(e) = \pm(i - j) \neq 0$ in $\mathbb{Z}_k$, so ϕ is nowhere-zero. $\square$

9.3. Berge-Fulkerson conjecture. An m -cycle k -cover is a list of m Eulerian subgraphs such that every edge belongs to exactly k of them.

Theorem 21 (Bermond, Jackson, and Jaeger [2]). *Every bridgeless graph has a 7-cycle 4-cover.*

Proof. It suffices to prove the claim for each connected component of G . Thus, after covering any loops twice, we may assume that G is 2-edge-connected. By Theorem 9, let C_1, C_2, C_3 be the supports of the three coordinates of a nowhere-zero $\mathbb{F}_2^3$ -flow. Every edge belongs to at least one of these Eulerian subgraphs. The seven nonzero linear combinations

$$C_1, C_2, C_3, C_1 \triangle C_2, C_2 \triangle C_3, C_1 \triangle C_3, C_1 \triangle C_2 \triangle C_3$$

form a 7-cycle 4-cover, because every nonzero vector of $\mathbb{F}_2^3$ has dot product 1 with exactly four of the seven nonzero vectors of $\mathbb{F}_2^3$. $\square$

By using the 6-flow theorem of Seymour [18], Fan [5] showed the following.

Theorem 22 (Fan [5]). *Every bridgeless graph admits a 10-cycle 6-cover.*

Conjecture 23 (Berge and Fulkerson [7]). *Every bridgeless cubic graph admits 6 perfect matchings such that every edge is in exactly two of them.*

If we take the complement of each perfect matching, we deduce the following equivalent form of the conjecture of Berge and Fulkerson.

Conjecture 24 (Berge and Fulkerson [7]). *Every bridgeless cubic graph admits a 6-cycle 4-cover.*

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